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TRAPEZOIDAL STRATIFIED MONTE CARLO INTEGRATION

by

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Abstract

Weighted integrals of random processes are approximated by the trapezoidal rule based on a stratified and symmetrized random sample of size n. The weight functions are assumed to be twice continuously differentiable. We consider the rate of convergence to zero of the mean-square integral approximation error as the sample size increases indefinitely. For random processes which are twice mean-square continuously differentiable it is shown that the rate is n^{-5}, just as without a random component (Haber [2]). For random processes which are a bit more than once, but not twice, mean-square continuously differentiable the rate is shown to be n^{-4}. In both cases the asymptotic constant is also determined.

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1. Introduction, Results, and Discussion

We consider the numerical approximation of the integral

$$I(f) = \int_0^1 f(t)\,dt \tag{1.1}$$

of a function f over a finite interval. The simple Monte Carlo numerical approximation of I takes the form

$$J_n^{(0)}(f) = \frac{1}{n} \sum_{i=1}^{n} f(U_i)$$

where $U_1, \cdots, U_n$ are independent random variables each with a uniform distribution over the interval $[0, 1]$. When f is square integrable the mean-square error is given by

$$E[I(f) - J_n^{(0)}(f)]^2 = \frac{1}{n} \{I(f^2) - [I(f)]^2\}$$

and the rate of n^{-1} cannot be improved by imposing additional smoothness assumptions on f.

Haber [1] introduced a stratified sampling scheme whereby the interval $[0, 1]$ is partitioned into n subintervals $A_{n,i}$, $i = 1, \cdots, n$, of equal length and a point $U_{n,i}$ is chosen at random, i.e., uniformly distributed, in $A_{n,i}$ (the $U_{n,i}$'s being independent for each n). Then the stratified Monte Carlo approximation of the integral $I(f)$ is

$$J_n^{(1)}(f) = \frac{1}{n} \sum_{i=1}^{n} f(U_{n,i}) .$$

When f has a continuous derivative on $[0, 1]$ the rate of quadratic-mean convergence is n^{-3} [1],

$$\lim_{n\to\infty} n^3 E[I(f) - J_n^{(1)}(f)]^2 = \frac{1}{12} \int_0^1 [f'(t)]^2 dt ,$$

and this rate cannot be improved by imposing further smoothness requirements on f. In order to obtain a faster rate of convergence when f has a continuous <u>second</u> derivative on $[0, 1]$, Haber [2] adopted the antithetic variates method and considered the following stratified and symmetrized scheme where along with each $U_{n,i}$ its antithetic point $U'_{n,i}$ (i.e., the symmetrically opposite point to $U_{n,i}$ in $A_{n,i}$) is used. The stratified and symmetrized Monte Carlo approximation of the integral $I(f)$ is

$$I_{2n}(f) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} [f(U_{n,i}) + f(U'_{n,i})] . \tag{1.2}$$

If f has a continuous second derivative on $[0,1]$ then the rate of convergence is n^{-5} and

$$\lim_{n \to \infty} (2n)^5 E[I(f) - I_{2n}(f)]^2 = \frac{2}{45} \int_0^1 [f''(t)]^2 \, dt . \tag{1.3}$$

In this paper we consider weighted integrals of random processes and establish the rate of quadratic-mean convergence and the asymptotic constant for estimates of the form (1.2), allowing for nonuniform partitions. Throughout this paper $X = \{X(t,\omega), 0 \le t \le 1\}$ is a measurable second-order random process with mean zero, $E[X(t)] = 0$, and covariance function $R(t,s) = E[X(t)X(s)]$, defined on a probability space (Ω, F, P). We shall be concerned with the numerical approximation of the integral

$$I(fX) = \int_0^1 f(t) X(t) \, dt \tag{1.4}$$

which exists as a sample path integral whenever $\int_0^1 |f(t)| R^{1/2}(t,t) \, dt < \infty$. (We suppress the probability variable ω and write $X(t)$ for $X(t,\omega)$). Integrals of the form (1.4) are common in detection and estimation problems. Unlike (1.2) we allow the partition $\{A_{n,i}\}_{i=1}^{n}$ of the interval $[0,1]$ to be nonequally-spaced and we adopt "regular" partitions $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = 1$ defined by means of continuous, strictly positive, probability density function $h(t)$ on $[0,1]$ as follows:

$$\int_0^{t_{n,i}} h(t) \, dt = \frac{i}{n} \quad ; \quad i = 0, 1, \cdots, n . \tag{1.5}$$

We set

$$A_{n,i} = (t_{n,i-1}, t_{n,i}), \quad , \quad \Delta t_{n,i} = t_{n,i} - t_{n,i-1} \quad , \quad i = 1, \cdots, n . \tag{1.6}$$

When $h(t) = 1$ we obtain a uniform partition of $[0,1]$. It will be seen subsequently that the quality of the approximation can be improved by tailoring the density $h(t)$ to the covariance $R(t,s)$ of the process X.

We assume that for each $n \ge 1$,

i) $U_{n,i}$ is uniformly distributed over $A_{n,i}$, $i = 1, \cdots, n$,

ii) $U_{n,1}, \cdots, U_{n,n}$ are independent,

iii) $\{U_{n,i}\}_{i=1}^{n}$ is independent of the process X.

We denote the antithetical point of $U_{n,i}$ by $U'_{n,i}$:

$$U'_{n,i} = 2c_{n,i} - U_{n,i} \tag{1.7}$$

where $c_{n,i}$ is the midpoint of $A_{n,i}$,

$$c_{n,i} = \frac{1}{2}(t_{n,i-1} + t_{ni}). \tag{1.8}$$

The stratified and symmetrized Monte Carlo approximation of the integral $I(fX)$ of (1.4) is now defined as

$$I_{2n}(fX) = \frac{1}{2}\sum_{i=1}^{n}\left\{f(U_{n,i})X(U_{n,i}) + f(U'_{n,i})X(U'_{n,i})\right\}\Delta t_{n,i} \tag{1.9}$$

and is in fact a trapezoidal rule. We first establish an expression for the quadratic-mean approximation error under general conditions. This is useful for evaluating finite sample size performance and for studying the asymptotic convergence properties.

<u>THEOREM 1.</u> If $\int_0^1 f^2(t)R(t,t)\,dt < \infty$ then for all $n \geq 1$ we have

$$E[I(fX) - I_{2n}(fX)]^2 = \sum_{i=1}^{n}\left\{\frac{1}{2}\Delta t_{n,i}\int_{A_{n,i}}[f^2(t)R(t,t) + f(t)R(t, 2c_{n,i} - t)f(2c_{n,i} - t)]\,dt\right.$$

$$\left. - \int_{A_{n,i}}\int_{A_{n,i}} f(t)R(t,s)f(s)\,dtds\right\}. \tag{1.10}$$

We next show that when the function f has a continuous second derivative and the process X has essentially one (but not two) quadratic-mean derivative which is mean-square continuous, then the rate of convergence of the quadratic-mean integral approximation error is precisely n^{-4} (<u>not</u> n^{-5} !). Specifically we make the following assumption.

ASSUMPTION A.

i) f has a continuous second derivative on $[0, 1]$.

ii) The covariance function $R(t, s)$ of the process X has continuous mixed derivatives $R^{k,j}(t, s)$ of order 2, $0 \le k+j \le 2$, on the unit square $[0, 1] \times [0, 1]$; and of order 3, $k+j=3$, off its diagonal with finite one-sided limits at the diagonal which are continuous along the diagonal.

iii) The function $r(t) = R(t, t)$ has a continuous third derivative on $[0, 1]$.

The assumption of continuous mixed derivatives of R of order up to 2 on $[0, 1] \times [0, 1]$ is equivalent to the assumption that the processes X has one mean-square continuous quadratic-mean derivative. The additional assumption of differentiability of order 3 off the diagonal is weak and is always satisfied when, for example, X is stationary, has rational spectral density, and exactly one quadratic-mean derivative. The smoothness assumption on $r(t)$ is very weak and is always satisfied in the stationary case. With

$$R_a^{k,j}(t, t) = \lim_{\substack{(u,v)\to(t,t) \\ u<v}} R^{k,j}(u,v), \quad R_b^{k,j}(t, t) = \lim_{\substack{(u,v)\to(t,t) \\ v<u}} R^{k,j}(u,v), \tag{1.11}$$

the one-sided limits of the derivatives of R above and below the diagonal, respectively, we set

$$\beta_{k,j}(t) = R_a^{k,j}(t, t) - R_b^{k,j}(t, t) \tag{1.12}$$

which exist under Assumption A for $k+j=3$. We can now state one of our main results.

THEOREM 2. Under Assumption A we have

$$\lim_{n\to\infty} (2n)^4 E[I(fX) - I_{2n}(fX)]^2 = \frac{1}{120} \int_0^1 \frac{f^2(t)}{h^4(t)} [3\beta_{3,0}(t) + 7\beta_{2,1}(t)]\, dt . \tag{1.13}$$

It is seen that the rate of quadratic-mean convergence is precisely n^{-4}, provided $3\beta_{3,0}(t) + 7\beta_{2,1}(t)$ is not identically zero, and cannot be improved by additional smoothness of f. In case the third order mixed derivatives of R are continuous at the diagonal, the asymptotic constant in (1.13) is zero and the mean-square approximation error is $o(n^{-4})$. The asymptotic constant in Theorem 2 depends on the *density function* h *of the regular partition.* The optimal density h^* which minimizes the asymptotic constant in (1.13) is given by

$$h^*(t) = \frac{\{f^2(t)[3\beta_{3,0}(t) + 7\beta_{2,1}(t)]\}^{2/5}}{\int_0^1 \{f^2(u)[3\beta_{3,0}(u) + 7\beta_{2,1}(u)]\}^{2/5}\, du} \tag{1.14}$$

for which (1.13) becomes

$$\lim_{n\to\infty} (2n)^4 E\,[I(fX) - I^*_{2n}(fX)]^2 = \frac{1}{120}\left\{\int_0^1 \left[f^2(t)[3\beta_{3,0}(t) + 7\beta_{2,1}(t)]\right]^{2/5} dt\right\}^5 .$$

When the process X is weakly-stationary, $R(t, s) = R(t - s)$, Assumption A simplifies to

ASSUMPTION A' (STATIONARY CASE).

i) f has a continuous second derivative on $[0, 1]$.

ii) The covariance function $R(t)$ has a continuous derivative of order 2 on the entire real line and of order 3 away from the origin with finite one-sided limits $R^{(3)}(\pm 0)$.

With

$$\beta_3 \triangleq R^{(3)}(0+) - R^{(3)}(0-) \geq 0\,, \tag{1.15}$$

Theorem 2 becomes

COROLLARY 1. When the process X is stationary, under Assumption A', we have

$$\lim_{n\to\infty} (2n)^4\, E\,[I(fX) - I_{2n}(fX)]^2 = \frac{\beta_3}{30} \int_0^1 \frac{f^2(t)}{h^4(t)}\, dt\,. \tag{1.16}$$

It is seen that if $R^{(3)}(t)$ is discontinuous at the origin, the quadratic-mean convergence rate is precisely n^{-4}; if $R^{(3)}(t)$ is continuous at the origin then $\beta_3 = 0$ and the mean-square approximation error is $o(n^{-4})$. The asymptotically optimal density is now given by

$$h^*(t) = \frac{|f(t)|^{2/5}}{\int_0^1 |f(u)|^{2/5}\, du}\,,$$

for which (1.16) becomes

$$\lim_{n\to\infty} (2n)^4\, E\,[I(fX) - I^*_{2n}(fX)]^2 = \frac{\beta_3}{30}\left\{\int_0^1 |f(t)|^{2/5}\, dt\right\}^5 .$$

It is clear from Theorem 2 that for the approximation of weighted integrals of random processes we obtain a quadratic-mean convergence rate of n^{-4} when the weight has two continuous derivatives and the process has essentially one but not quite two mean-square continuous quadratic-mean derivatives. We now show that under an additional smoothness condition on the covariance function $R(t, s)$ of the process X, we can obtain a rate of n^{-5} for weighted integrals of random processes. To this end we set

ASSUMPTION B.

i) f has a continuous second derivative on $[0, 1]$.

ii) The covariance function $R(t, s)$ of the process X has continuous mixed derivatives $R^{k,j}(t, s)$ of order 4, $0 \le k + j \le 4$, on the unit square $[0, 1] \times [0, 1]$.

Part (ii) of Assumption B is equivalent to the assumption that the process X has two mean-square continuous quadratic-mean derivatives. We then have our second principal result.

THEOREM 3. Under Assumption B we have

$$\lim_{n \to \infty} (2n)^5 \, E[I(fX) - I_{2n}(fX)]^2 = \frac{2}{45} \int_0^1 \frac{A^2(t)}{h^5(t)} \, dt \tag{1.17}$$

where

$$\begin{aligned} A^2(t) &= R(t, t)[f''(t)]^2 + 4R^{1,0}(t, t)f'(t)f''(t) \\ &\quad + 4\left[\frac{1}{2}R^{2,0}(t, t)f(t)f''(t) + R^{1,1}(t, t)(f'(t))^2\right] + 4R^{2,1}(t, t)f(t)f'(t) + R^{2,2}(t, t)f^2(t) \\ &= E\{[f(t)X(t)]''\}^2 \end{aligned} \tag{1.18}$$

and differentiation of the process X is meant in quadratic-mean.

Since $A^2(t)$ cannot be identically zero, the rate of quadratic-mean convergence is precisely n^{-5} and cannot be improved by additional smoothness of f or R. As in the discussion following Theorem 2, we can select the partitioning density h so as to minimize the asymptotic constant in Theorem 3. We obtain

$$h^*(t) = \frac{|A(t)|^{2/6}}{\int_0^1 |A(u)|^{2/6} du} ,$$

for which (1.17) becomes

$$\lim_{n\to\infty} (2n)^5\, E\,[I(fX) - I^*_{2n}(fX)]^2 = \frac{2}{45}\left\{\int_0^1 |A(t)|^{2/6}\, dt\right\}^6 .$$

We now specialize Theorem 3 to the stationary case. Here Assumption B simplies to the following.

ASSUMPTION B' (STATIONARY CASE).

i) f has a continuous second derivative on $[0, 1]$.

ii) The covariance function $R(t)$ has a continuous derivative of order 4.

We then have

COROLLARY 2. When the process X is stationary, under Assumption B', we have

$$\lim_{n\to\infty} (2n)^5\, E\,[I(fX) - I_{2n}(fX)]^2 = \frac{2}{45}\int_0^1 \frac{\bar{A}^2(t)}{h^5(t)}\, dt \tag{1.19}$$

where

$$\begin{aligned}\bar{A}^2(t) &= R(0)[f''(t)]^2 + 2[-R''(0)]\,\{2[f'(t)]^2 - f(t)f''(t)\} + R^{(4)}(0)\, f^2(t) \\ &= E\,\{[f(t)X(t)]''\}^2 .\end{aligned} \tag{1.20}$$

The asymptotically optimal partitioning density is now given by

$$h^*(t) = \frac{\left[\bar{A}^2(t)\right]^{1/6}}{\int_0^1 [\bar{A}^2(u)]^{1/6}\, du} .$$

In the stratified and symmetrized Monte Carlo approximation considered in this paper, the randomly chosen points $\{U_{n,i}\}_{i=1}^n$ are uniformly distributed within each subinterval of the regular partition. On the other hand one may wish to retain the property of the crude Monte Carlo whereby the randomly chosen points $\{U_{n,i}\}_{i=1}^n$ are uniformly distributed over the domain of integration $[0, 1]$. Such an approach leads to trapezoidal Monte Carlo integration which was considered in Yakowitz et al. [4] for integration of (deterministic) functions f and in Masry and Cambanis [3] for weighted integrals of random processes. It

may be of interest to provide a comparison of the performance of these two integral approximation schemes (both of which use trapezoidal rules) under identical assumptions on the integrands. For the trapezoidal Monte Carlo approximation we use independent random variables $U_1, \cdots, U_n$ uniformly distributed on [0, 1], independent of X, and we let $\tau_{n,0} \triangleq 0 < \tau_{n,1} < \tau_{n,2} < \cdots < \tau_{n,n} < 1 \triangleq \tau_{n,n+1}$ be the corresponding ordered sample. The integral (1.4) is approximated by

$$I_{n+2}^{(\mathrm{trap})}(fX) = \frac{1}{2} \sum_{i=0}^{n} \Big[f(\tau_{n,i}) X(\tau_{n,i}) + f(\tau_{n,i+1}) X(\tau_{n,i+1}) \Big] (\tau_{n,i+1} - \tau_{n,i}) . \tag{1.21}$$

For simplicity we state below the convergence properties of $I_{n+2}^{(\mathrm{trap})}$ in the stationary case only. Under Assumption A' we have [3]

$$\lim_{n \to \infty} n^4 E\,[I(fX) - I_{n+2}^{(\mathrm{trap})}(fX)]^2 = \frac{3}{4} \beta_3 \int_0^1 f^2(t) dt + \frac{1}{4} E[(fX)'(1) - (fX)'(0)]^2 \equiv C_{\mathrm{trap}} .$$

In order to compare this to Corollary 1, we assume even sample size N so that

$$\lim_{N \to \infty} N^4 \, E\Big[I(fX) - I_N^{(\mathrm{trap})}(fX) \Big]^2 = C_{\mathrm{trap}}$$

whereas by Corollary 1 with $h(t) = 1$ we have

$$\lim_{N \to \infty} N^4 \, E\Big[I(fX) - I_N(fX) \Big]^2 = \frac{\beta_3}{30} \int_0^1 f^2(t)\, dt \equiv C_{\mathrm{str}} .$$

It is clear that, while the symmetric-stratified and the trapezoidal Monte Carlo approximations have identical rates of quadratic-mean convergence, their corresponding asymptotic constants satisfy $(C_{\mathrm{trap}} / C_{\mathrm{str}})^{1/4} > (45/2)^{1/4} = 2.18$ and thus, asymptotically, for the same accuracy measured in terms of quadratic-mean error, more than twice as many samples are required for the trapezoidal scheme. This discrepancy also appears in the example below where the finite sample size performance is evaluated.

Finally, it may be interest to examine the performance of the stratified-symmetrized Monte Carlo integral approximation when the function f and the process X satisfy weaker smoothness conditions than those stated earlier. For integrals of random process $I(fX)$ we assume for simplicity that $f \equiv 1$ and X is wide-sense stationary process. The following table of quadratic-mean convergence rates complements Theorems 2 and 3. The additional rates displayed in the table can be established in the manner of the

proofs of Theorems 2 and 3.

$I_{2n}(f)$		$I_{2n}(X)$	
Smoothness	Rate	Smoothness	Rate
f continuous	$o(n^{-1})$	R continuous	$o(n^{-1})$
		R' continuous	$o(n^{-2})$
f' continuous	$o(n^{-3})$	R'' continuous	$o(n^{-3})$
		R'' continuous & $R^{(3)}(0\pm)$ finite, $\neq 0$	n^{-4}
f'' continuous	n^{-5}	$R^{(4)}$ continuous	n^{-5}

Recall that a wide-sense stationary process X has k mean-square continuous quadratic-mean derivatives if and only if $R^{(2k)}$ is continuous. It is then seen from the table that when the nonrandom function f or the stationary process X have 0, 1, or 2 derivatives, usual or quadratic-mean respectively, the rates of convergence of the mean-square approximation error of their integrals are identical. For the approximation $I_{2n}(fX)$ of $I(fX)$ with mixed smoothness conditions on f and on X, it can be shown that the slower rate prevails. Thus, for example, if f' is continuous and $R^{(4)}$ is continuous, the rate of convergence of the mean-square approximation is $o(n^{-3})$.

Thus the ultimate rate of convergence, n^{-5}, of the symmetric-stratified Monte Carlo approximation of $I(fX)$ is achieved when the nonrandom function f has two continuous derivatives and the random process X has two mean-square continuous quadratic-mean derivatives; i.e. when the usual smoothness of f and the quadratic-mean smoothness of X are comparable. This is in contrast with the trapezoidal Monte Carlo approximation of $I(fX)$ whose ultimate rate, n^{-4}, is achieved when the nonrandom function f has two continuous derivatives and the random process X has one mean-square continuous quadratic-mean derivative and continuous mixed partial derivatives of R of order 3 off the diagonal, but not two quadratic-mean derivatives; i.e., when the quadratic-mean smoothness of X is less than the usual smoothness of f!

EXAMPLE. We illustrate via an example the finite sample size performance of the stratified-symmetrized Monte Carlo approximation and compare it to that of the trapezoidal Monte Carlo approximation. We consider a stationary process X with mean zero and covariance function

$$R(t) = (1 + \gamma |t|)\, e^{-\gamma |t|}$$

where $\gamma > 0$. Note that $R(0) = 1$ and the process has exactly one quadratic mean derivative. Moreover, $R^{(3)}(t)$ is discontinuous at the origin so that Corollary 1 is applicable and the rate of mean-square error convergence of $I_{2n}(fX)$ is n^{-4}. For simplicity we take $f(t) \equiv 1$ so that the integral to be estimated is $I(X) = \int_0^1 X(t)\,dt$. We choose an equally-spaced partition, $h(t) \equiv 1$, for which the approximation (1.9) becomes

$$I_{2n}(X) = \frac{1}{2n} \sum_{i=1}^{n} \left\{ X(U_{n,i}) + X(U'_{n,i}) \right\}.$$

The variance σ^2 of $I(X)$ is given by

$$\sigma^2 = E[(X)]^2 = \int_0^1\!\!\int_0^1 R(t-s)\,dt\,ds = \frac{2}{\gamma}\left\{ 2 - \frac{3}{\gamma} + \left[1 + \frac{3}{\gamma} \right] e^{-\gamma} \right\}.$$

From Theorem 1 we find after some algebra, that the mean-square error is given by

$$E[I(X) - I_{2n}(X)]^2 = \left[\frac{1}{2n} + \frac{6n}{\gamma^2} \right] \left[1 - e^{-\gamma/n} \right] - \frac{3}{\gamma} \left[1 + e^{-\gamma/n} \right].$$

The asymptotic constant C_{str} is given by

$$C_{str} = \frac{\gamma^3}{120}.$$

Let $N = 2n = 2, 4, \cdots,$ be the (true) sample size with corresponding mean-square error $mse(N) = E[I(X) - I_N(X)]^2$. The fractional mean-square error is then given by $mse(N)/\sigma^2$. In order to select appropriate values of γ for numerical display of the finite sample size performance, the behavior of the fractional error $mse(2)/\sigma^2$ (based on 2 samples) as a function of γ was investigated. Table 1 below lists the results along with the value of the asymptotic constant.

γ	$mse(2)/\sigma^2$	C_{str}
.2	6.055×10^{-5}	6.66×10^{-5}
1	5.456×10^{-3}	8.33×10^{-3}
3	8.02×10^{-2}	.225
5	.2321	1.04
7	.4293	2.858
10	.7646	8.333
15	1.361	28.125
20	1.973	66.666

We select two values $\gamma = 5$ and $\gamma = 10$ corresponding to moderate values of $mse(2)/\sigma^2$.

In Figure 1 the fractional mean-square error $mse(N)/\sigma^2$ is plotted as a function of the sample size $N = 2, 4, 6, \cdots, 30$ for $\gamma = 5$ and $\gamma = 10$. It is seen that for the smaller value of $\gamma = 5$, the fractional error is considerably smaller for each sample size N. This can be explained by the less rapid decay of $R(t)$ and hence the larger correlation between consecutive samples so that $I_N(X)$ provides a better estimate of $I(X)$ in this case. The closeness of the fractional mean-square error to its asymptotic value,

$$mse(N)/\sigma^2 \sim \frac{C_{str}/\sigma^2}{N^4},$$

is displayed in Figure 2 for parameter $\gamma = 5$. Note that the asymptotic value overestimates the true error for all sample sizes N in the plotted range. Naturally, the discrepancy between the two values diminishes as N increases.

It may be of interest to compare the above finite sample size performance to that of the trapezoidal Monte Carlo approximation (1.21). For the latter approximation the expression for the mean-square error for a finite sample size is given in [3, Eq. (1.13)]. In Figure 3, the fractional mean-square errors $mse(N)/\sigma^2$ are plotted as functions of the sample size $N = 2, 4, \cdots, 30$ for $\gamma = 5$. It is seen that the symmetric-stratified approximation outperforms the trapezoidal Monte Carlo approximation by a wide margin for all sample sizes N in the plotted range. For 1% fractional mean-square error 6 samples are required for the symmetric-stratified approximation but 12 samples for the trapezoidal Monte Carlo approximation; for .1% fractional mean-square error, the corresponding numbers of samples are 12 and

24 respectively. Asymptotically, for large N, we can compare instead the asymptotic constants

$$C_{\rm trap} = \{1 + 6\gamma + (\gamma - 1)\, e^{-\gamma}\}\, \gamma^2/2\,, \quad C_{\rm str} = \gamma^3/120$$

and it is seen that for all $\gamma > 0$,

$$C_{\rm trap}/C_{\rm str} > 360$$

so that, for the same mean-square error, the trapezoidal Monte Carlo approximation requires a sample size N greater than that of the symmetric-stratified Monte Carlo approximation by a factor of at least $(360)^{1/4} = 4.35$. When $\gamma = 5$, $[C_{\rm trap} / C_{\rm str}]^{1/4} = (372.32)^{1/4} = 4.393$.

2. Derivations

In order to simplify the writing throughout this section we will drop the subscript n from $A_{n,i}$, $t_{n,i}$, $\Delta t_{n,i}$, $c_{n,i}$, $U_{n,i}$.

PROOF OF THEOREM 1. The expectation in $E[I(fX) - I_{2n}(fX)]^2$ is with respect to both the random samples $\{U_i\}_{i=1}^n$ and the random process $\{X(t), 0 \le t \le 1\}$ which are mutually independent. We first verify that both $I(fX)$ and $I_{2n}(fX)$ have finite second moments. To simplify the notation we put $Y(t) = f(t)X(t)$ and $M(t, s) = f(t)\, R(t, s) f(s)$. It follows that

$$E\,\{\,|\,I(Y)|^2\} \le E\{\int_0^1 E\{\,|Y(t)|\,dt\}^2 = \int_0^1\int_0^1 E\{\,|\,Y(t)Y(s)|\,\}\, dtds \le \{\int_0^1 M^{1/2}(t,t)\, dt\,\}^2 < \infty$$

where we used $E^2\{\,|\,Y(t)Y(s)|\,\} \le E\{Y^2(t)\}\; E\,\{Y^2(s)\} = M(t, t)\, M(s, s)$. The more restrictive condition $\int_0^1 M(t, t)\, dt < \infty$ is needed for the finiteness of the second moment of $I_{2n}(Y)$. Indeed, taking first the expectation with respect to the random samples, we find for each i,

$$E\{Y^2(U_i)\} = E\left\{\frac{1}{\Delta t_i}\int_{A_i} Y^2(t) dt\right\} = \frac{1}{\Delta t_i}\int_{A_i} M(t, t)\, dt < \infty\,.$$

These inequalities justify the interchanging of integrals and expectations below. In view of (1.1) and (1.9) the integral approximation error can be written as

$$I(Y) - I_{2n}(Y) = \sum_{i=1}^{n} \left\{ \int_{Ai} Y(t)\,dt - \frac{1}{2}\Delta t_i [Y(U_i) + Y(2c_i - U_i)] \right\} \triangleq \sum_{i=1}^{n} e_i \,.$$

The bias of the ith error term is

$$E\{e_i\} = E\{E(e_i \mid X)\} = E\left\{ \int_{A_i} Y(t)\,dt - \frac{1}{2} \int_{A_i} [Y(t) + Y(2c_i - t)]\,dt \right\} = E\{0\} = 0 \,.$$

Thus I_{2n} is an unbiased estimator of I:

$$E\{I(Y) - I_{2n}(Y)\} = 0 \,.$$

Since given the random process X, the error terms $\{e_i\}_{i=1}^{n}$ are independent with zero mean we have for $i \neq j$,

$$E\{e_i e_j\} = E\{E(e_i e_j \mid X)\} = E\{E(e_i \mid X) E(e_j \mid X)\} = E(0) = 0 \,.$$

It follows that

$$mse_{2n} \triangleq E[I(Y) - I_{2n}(Y)]^2 = E\left\{ \sum_{i=1}^{n} e_i \right\}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} E\{e_i e_j\} = \sum_{i=1}^{n} E\{e_i^2\} \,.$$

Performing first the expectation with respect to the samples we find

$$E\{e_i^2\} = E\left\{ \int_{A_i} Y(t)\,dt - \frac{1}{2}\Delta t_i [Y(U_i) + Y(2c_i - U_i)] \right\}^2$$

$$= E\left\{ \left[\int_{A_i} Y(t)dt \right]^2 - \Delta t_i \int_{A_i} Y(t)dt\, [Y(U_i) + Y(2c_i - U_i)] + \frac{1}{4}\Delta t_i^2 [Y(U_i) + Y(2c_i - U_i)]^2 \right\}$$

$$= E\left\{ \left[\int_{A_i} Y(t)dt \right]^2 - \int_{Ai} Y(t)dt \int_{A_i} [Y(s) + Y(2c_i - s)]ds + \frac{1}{4}\Delta t_i \int_{A_i} [Y(t) + Y(2c_i - t)]^2\,dt \right\}$$

$$= \int_{A_i}\int_{A_i} M(t, s)\,dtds - \int_{A_i}\int_{A_i} [M(t, s) + M(t, 2c_i - s)]\,dtds$$

$$+ \frac{1}{4}\Delta t_i \int_{A_i} [M(t, t) + 2M(t, 2c_i - t) + M(2c_i - t, 2c_i - t)]\,dt$$

$$= \frac{1}{2}\,\Delta t_i \int_{A_i} [M(t,\,t) + M(t,\,2c_i - t)]\,dt - \int_{A_i}\int_{A_i} M(t,\,s)\,dtds\ .$$

and (1.10) follows by summation. □

<u>PROOF OF THEOREM 2.</u> We first Taylor-expand $R(t,s)$ for $(t,\,s)$ off the diagonal of $A_i \times A_i$ (i.e., $t \neq s$) about the center $(c_i,\,c_i)$. From Assumption $A.(ii)$ we have

$$\begin{aligned} R(t,\,s) &= R(c_i,\,c_i) + (t - c_i)\,R^{1,0}(c_i\,,c_i) + (s - c_i)\,R^{0,1}(c_i,\,c_i) \\ &+ \frac{1}{2}(t - c_i)^2\,R^{2,0}(c_i,\,c_i) + \frac{1}{2}(s - c_i)^2\,R^{0,2}(c_i,\,c_i) + (t - c_i)(s - c_i)\,R^{1,1}(c_i,\,c_i) \\ &+ \frac{1}{6}(t - c_i)^3\,R^{3,0}(\text{ int }) + \frac{1}{2}(t - c_i)^2(s - c_i)R^{2,1}(\text{ int }) \\ &+ \frac{1}{2}(t - c_i)(s - c_i)^2\,R^{1,2}(\text{ int }) + \frac{1}{6}(s - c_i)^3\,R^{0,3}(\text{ int }) \end{aligned} \tag{2.1}$$

where int is a point in the open line segment determined by $(t,\,s)$ and $(c_i,\,c_i)$ (depending of course on both). We also Taylor-expand the function $r(t) = R(t,\,t)$ about c_i. In view of Assumption $A.(ii) - (iii)$ we have $r'(t) = 2\,R^{1,0}(t,\,t)$, $r''(t) = 2[R^{2,0}(t,\,t) + R^{1,1}(t,\,t)]$ and thus

$$\begin{aligned} R(t,\,t) &= R(c_i,\,c_i) + (t - c_i)2\,R^{1,0}(c_i,\,c_i) + (t - c_i)^2[R^{2,0}(c_i,\,c_i) + R^{1,1}(c_i,\,c_i)] \\ &+ \frac{1}{6}(t - c_i)^3\,r'''(\text{ int }) \end{aligned} \tag{2.2}$$

where int is a point in between t and c_i (depending on both). Substituting (2.1) and (2.2) into (1.10) and regrouping terms using the symmetry of $R(t,\,s)$ we obtain

$$MSE_n = E[I(fX) - I_{2n}(fX)]^2 \tag{2.3}$$

$$= \sum_{i=1}^{n} R(c_i,\,c_i)\left\{\frac{1}{2}\Delta t_i \int_{A_i} f^2 + \frac{1}{2}\Delta t_i \int_{A_i} f(t) f(2c_i - t)\,dt - (\int_{A_i} f)^2\right\} \qquad \triangleq E_n^{0,0} \tag{2.3.1}$$

$$+ \sum_{i=}^{n} R^{1,0}(c_i,\,c_i)\left\{\Delta t_i \int_{A_i} (t - c_i) f^2(t)\,dt - 2(\int_{A_i} f) \int_{A_i} (t - c_i) f(t)\,dt\right\} \qquad \triangleq E_n^{1,0} \tag{2.3.2}$$

$$+ \sum_{i=1}^{n} R^{2,0}(c_i,c_i)\left\{\frac{1}{2}\Delta t_i \int_{A_i} (t - c_i)^2 f^2(t)\,dt + \frac{1}{2}\Delta t_i \int_{A_i} (t - c_i)^2 f(t) f(2c_i - t)\,dt\right.$$

$$-\left(\int_{A_i} f\right)\int_{A_i}(t-c_i)^2 f(t)\,dt\Bigg\} \qquad \triangleq E_n^{2,0} \qquad (2.3.3)$$

$$+\sum_{i=1}^{n} R^{1,1}(c_i,c_i)\Bigg\{\frac{1}{2}\Delta t_i\int_{A_i}(t-c_i)^2 f^2(t)\,dt-\frac{1}{2}\Delta t_i\int_{A_i}(t-c_i)^2 f(t)f(2c_i-t)\,dt$$

$$-\left(\int_{A_i}(t-c_i)f(t)\,dt\right)^2\Bigg\} \qquad \triangleq E_n^{1,1} \qquad (2.3.4)$$

$$+\sum_{i=1}^{n}\frac{1}{12}\Delta t_i\int_{A_i}(t-c_i)^3 f^2(t)r'''(\text{int})\,dt \qquad \triangleq E_n^{r} \qquad (2.3.5)$$

$$+\sum_{i=1}^{n}\Bigg\{\frac{1}{2}\Delta t_i\int_{A_i} f(t)f(2c_i-t)\Big[\frac{1}{6}(t-c_i)^3 R^{3,0}(u_t,2c_i-u_t)+\frac{1}{6}(c_i-t)^3 R^{0,3}(u_t,2c_i-u_t)$$

$$+\frac{1}{2}(t-c_i)^2(c_i-t)R^{2,1}(u_t,2c_i-u_t)+\frac{1}{2}(t-c_i)(c_i-t)^2 R^{1,2}(u_t,2c_i-u_t)\Big]\,dt$$

$$-\iint_{\substack{A_iA_i\\ t\neq s}} f(t)f(s)\Big[\frac{1}{6}(t-c_i)^3 R^{3,0}(\text{int})+\frac{1}{6}(s-c_i)^3 R^{0,3}(\text{int}) \qquad (2.3.6)$$

$$+\frac{1}{2}(t-c_i)^2(s-c_i)R^{2,1}(\text{int})+\frac{1}{2}(t-c_i)(s_i-c_i)^2 R^{1,2}(\text{int})\Big]\,dtds\Bigg\} \qquad \triangleq E_n^{3}$$

In (2.3.5), int denotes a point in A_i depending on t. In the first term of (2.3.6), $R(t, 2c_i - t)$ is expanded about $R(c_i\ c_i)$ and thus the point u_t is in between t and c_i, and in the second term the intermediate point " int " is in between (t, s) and (c_i, c_i) and we excluded the diagonal $t = s$ from the integration because it has zero Lebesgue measure.

We now use the Taylor expansion of $f(t)$ about c_i, which in view of Assumption $A.(i)$ has the form

$$f(t)=f(c_i)+(t-c_i)f'(c_i)+\frac{1}{2}(t-c_i)^2 f''[d(t,c_i)] \qquad (2.4)$$

and the intermediate point d depends on t and c_i.

<u>Terms involving $R(c_i, c_i)$ $(E_n^{0,0})$.</u> From (2.4) we find

$$\int_{A_i} f=\Delta t_i f(c_i)+\Delta t_i^3\frac{1}{24}f''(\text{int}_1), \qquad (2.5)$$

$$\int_{A_i} f^2=\Delta t_i f^2(c_i)+\Delta t_i^3\frac{1}{12}\{f(c_i)f''(\text{int}_2)+[f'(c_i)]^2\}$$

$$+ \Delta t_i^4 \frac{1}{64} f'(c_i)[f''(\text{int}_4) - f''(\text{int}_3)] + \Delta t_i^5 \frac{1}{320} [f''(\text{int}_2)]^2 , \tag{2.6}$$

$$\int_{A_i} f(t) f(2c_i - t)\, dt = \Delta t_i f^2(c_i) + \Delta t_i^3 \frac{1}{12} \{f(c_i) f''(\text{int}_1) - [f'(c_i)]^2\}$$

$$+ \Delta t_i^4 \frac{1}{64} f'(c_i)[f''(\text{int}_3) - f''(\text{int}_4)] + \Delta t_i^5 \frac{1}{320} f''(\text{int}_5) f''(2c_i - \text{int}_5) , \tag{2.7}$$

and substituting into (2.3.1) we obtain

$$E_n^{0,0} = \sum_{i=1}^{n} R(c_i, c_i) \Delta t_i^6 \frac{1}{64} \left\{ \frac{1}{10} [f''(\text{int}_2)]^2 + \frac{1}{10} f''(\text{int}_5) f''(2c_i - \text{int}_5) - \frac{1}{9} [f''(\text{int}_1)]^2 \right\} . \tag{2.8}$$

Using

$$\frac{1}{n} = \int_{A_i} h(t) dt = h(\text{int}) \Delta t_i \tag{2.9}$$

it follows from (2.8) by Riemann integrability that as $n \to \infty$,

$$(2n)^5 E_n^{0,0} \to \frac{2}{45} \int_0^1 \frac{R(t,t)[f''(t)]^2}{h^5(t)}\, dt . \tag{2.10}$$

<u>Terms involving $R^{1,0}(c_i, c_i)$ ($E_n^{1,0}$).</u> From (2.4) we find

$$\int_{A_i} (t - c_i) f(t)\, dt = \Delta t_i^3 \frac{1}{12} f'(c_i) + \Delta t_i^4 \frac{1}{128} [f''(\text{int}_4) - f''(\text{int}_3)] , \tag{2.11}$$

$$\int_{A_i} (t - c_i) f^2(t)\, dt = \Delta t_i^3 \frac{1}{6} f(c_i) f'(c_i) + \Delta t_i^4 \frac{1}{64} f(c_i)[f''(\text{int}_4) - f''(\text{int}_3)] ,$$

$$+ \Delta t_i^5 \frac{1}{80} f'(c_i) f''(\text{int}_6) + \Delta t_i^6 \frac{1}{1536} \{[f''(\text{int}_8)]^2 - [f''(\text{int}_3)]^2\} , \tag{2.12}$$

and substituting them, along with (2.5), into (2.3.2) we obtain

$$E_n^{1,0} = \sum_{i=1}^{n} R^{1,0}(c_i, c_i) \left\{ \Delta t_i^6 \frac{1}{144} f'(c_i)[\frac{9}{5} f''(\text{int}_6) - f''(\text{int}_1)] + o(n^{-7}) \right\}$$

where the term $o(n^{-7})$ is uniform in i since by (2.9), $\Delta t_i \le (\varepsilon n)^{-1}$ for some $\varepsilon > 0$ as h is bounded away from zero. It then follows from (2.9) that as $n \to \infty$,

$$(2n)^5 E_n^{1,0} \to \frac{8}{45} \int_0^1 \frac{R^{1,0}(t,t) f'(t) f''(t)}{h^5(t)} \, dt \, . \tag{2.13}$$

Terms involving $R^{2,0}(c_i, c_i)$ ($E_n^{2,0}$). From (2.4) we find

$$\int_{A_i} (t - c_i)^2 f(t)\, dt = \Delta t_i^3 \frac{1}{12} f(c_i) + \Delta t_i^5 \frac{1}{160} f''(\mathrm{int}_6)\, , \tag{2.14}$$

$$\int_{A_i} (t - c_i)^2 f^2(t)\, dt = \Delta t_i^3 \frac{1}{12} f^2(c_i) + \Delta t_i^5 \frac{1}{80} \{[f'(c_i)]^2 + f(c_i) f''(\mathrm{int}_6)\}$$

$$+ \Delta t_i^6 \frac{1}{384} f'(c_i)[f''(\mathrm{int}_{11}) - f''(\mathrm{int}_{10})] + \Delta t_i^7 \frac{1}{1792} [f''(\mathrm{int}_9)]^2\, , \tag{2.15}$$

$$\int_{A_i} (t - c_i)^2 f(t) f(2c_i - t)\, dt = \Delta t_i^3 \frac{1}{12} f^2(c_i) + \Delta t_i^5 \frac{1}{80} \{f(c_i) f''(\mathrm{int}_6) - [f'(c_i)]^2\}$$

$$+ \Delta t_i^6 \frac{1}{384} f'(c_i)[f''(\mathrm{int}_{10}) - f''(\mathrm{int}_{11})] + \Delta t_i^7 \frac{1}{1792} f''(\mathrm{int}_{12}) f''(2c_i - \mathrm{int}_{12})\, , \tag{2.16}$$

and substituting into (2.3.3) along with (2.5) we obtain

$$E_n^{2,0} = \sum_{i=1}^{n} R^{2,0}(c_i, c_i) \left\{ \Delta t_i^6 \frac{1}{32} f(c_i) [\frac{1}{5} f''(\mathrm{int}_6) - \frac{1}{9} f''(\mathrm{int}_1)] + o(n^{-6}) \right\}$$

where the $o(n^{-6})$ term is uniform in i. It follows from (2.9) that as $n \to \infty$,

$$(2n)^5 E_n^{2,0} \to \frac{4}{45} \int_0^1 \frac{R^{2,0}(t,t) f(t) f''(t)}{h^5(t)} \, dt \, . \tag{2.17}$$

Terms involving $R^{1,1}(c_i, c_i)$ ($E_n^{1,1}$). Substituting (2.11), (2.15) and (2.16) into (2.3.4) we obtain

$$E_n^{1,1} = \sum_{i=1}^{n} R^{1,1}(c_i, c_i) \left\{ \Delta t_i^6 \frac{1}{180} [f^1(c_i)]^2 + o(n^{-6}) \right\}$$

where the $o(n^{-6})$ term is uniform in i, and thus by (2.9), as $n \to \infty$,

$$(2n)^5 E_n^{1,1} \to \frac{8}{45} \int_0^1 \frac{R^{1,1}(t,t) [f'(t)]^2}{h^5(t)} \, dt \, . \tag{2.18}$$

Terms involving r''' (E_n^r). Substituting (2.4) into (2.35) we obtain

$$E_n^r = \frac{1}{12} \sum_{i=1}^{n} \left\{ \Delta t_i^5 \frac{1}{64} f^2(c_i) [r'''(\mathrm{int}_2) - r'''(\mathrm{int}_1)] + o(n^{-5}) \right\}$$

where the $o(n^{-5})$ term is uniform in i, and using (2.9) we find as $n \to \infty$,

$$(2n)^4 E_n^r \to \frac{1}{12\cdot 64} \int_0^1 \frac{f^2(t)}{h^4(t)} [r'''(t) - r'''(t)]\, dt = 0 . \tag{2.19}$$

Terms involving $R^{3,0}, R^{2,1}$ (E_n^3). When we substitute (2.4) into (2.3.6) the dominant term corresponds to $f(c_i)$. We thus have

$$E_n^3 = \sum_{i=1}^{n} [f^2(c_i) + o(1)] \times \tag{2.20}$$

$$\times \Bigg\{ \frac{1}{4} \Delta t_i \int_{A_i} (t - c_i)^3 [\frac{1}{3} R^{3,0}(u_t, 2c_i - u_t) - \frac{1}{3} R^{0,3}(u_t, 2c_i - u_t)$$

$$- R^{2,1}(u_t, 2c_i - u_t) + R^{1,2}(u_t, 2c_i - u_t)]\, dt \tag{2.20.1}$$

$$- \iint_{\substack{A_i A_i \\ t \neq s}} [\frac{1}{6} (t - c_i)^3 R^{3,0}(\text{ int }) + \frac{1}{6} (s - c_i)^3 R^{0,3}(\text{ int })$$

$$+ \frac{1}{2} (t - c_i)^2 (s - c_i) R^{2,1}(\text{ int }) + \frac{1}{2} (t - c_i)(s - c_i)^2 R^{1,2}(\text{ int })]\, dt\, ds \Bigg\} . \tag{2.20.2}$$

where $o(1)$ is uniform in i. In view of the possible discontinuity of the third order mixed partial derivatives of R at the diagonal (Assumption $A.(ii)$), we proceed as follows. For the first term (2.20.1) in (2.20), we split the integral $\int_{A_i}$ into the two parts $\int_{t_{i-1}}^{c_i} + \int_{c_i}^{t_i}$ and then apply the mean value theorem to each part, since $(t - c_i)^3$ has constant sign over each half-interval (t_{i-1}, c_i), (c_i, t_i), to obtain, e.g.,

$$\int_{A_i} (t - c_i)^3 R^{3,0}(u_t, 2c_i - u_t)\, dt = R^{3,0}(a_1) \int_{t_{i-1}}^{c_i} (t - c_i)^3\, dt + R^{3,0}(b_1) \int_{c_i}^{t_i} (t - c_i)^3\, dt$$

$$= \frac{1}{64} \Delta t_i^4 [-R^{3,0}(a_1) + R^{3,0}(b_1)] ,$$

where a_1 and b_1 denote intermediate points in $A_i \times A_i$ above and below its diagonal respectively. (For $R(t, s)$, above the diagonal means $t < s$, and below the diagonal means $t > s$). For the second term (2.20.2) of (2.20) we split each of the four double integrals $\iint_{\substack{A_i \times A_i \\ t \neq s}}$ into four parts corresponding to the

regions above and below the diagonal where the terms $(t-c_i)^3$, $(s-c_i)^3$, $(t-c_i)^2(s-c_i)$, $(t-c_i)(s-c_i)^2$ have constant sign, and then apply the mean value theorem. For instance we get

$$\iint_{A_iA_i} (t-c_i)^3\, R^{3,0}(\text{ int })dt\,ds = \left\{ \iint_{\substack{t<s<t_i \\ t_{i-1}<t<c_i}} + \iint_{c_i<t<s<t_i} + \iint_{\substack{t_{i-1}<s<t \\ c_i<t<t_i}} + \iint_{t_{i-1}<s<t<c_i} \right\} (t-c_i)^3\, R^{3,0}(\text{ int })\, dt\, ds$$

$$= \left\{ -R^{3,0}(a_5)\Delta t_i^5\,\frac{9}{640} + R^{3,0}(a_6)\Delta t_i^5\,\frac{1}{640} + R^{3,0}(b_5)\Delta t_i^5\,\frac{9}{640} - R^{3,0}(b_6)\Delta t_i^5\,\frac{1}{640} \right\}$$

$$= \Delta t_i^5\,\frac{1}{640}\left\{ -9R^{3,0}(a_5) + R^{3,0}(a_6) + 9R^{3,0}(b_5) - R^{3,0}(b_6) \right\}$$

where a_k and b_k denote intermediate points above and below the diagonal in $A_i \times A_i$ respectively. Proceeding likewise for the remaining terms we obtain

$$E_n^3 = \sum_{i=1}^{n} [f^2(c_i) + o(1)]\,\Delta t_i^5 \times$$

$$\times \left\{ \frac{1}{256}\left[-\frac{1}{3}R^{3,0}(a_i) + \frac{1}{3}R^{3,0}(b_1) + \frac{1}{3}R^{0,3}(a_2) - \frac{1}{3}R^{0,3}(b_2) \right.\right.$$

$$\left. + R^{2,1}(a_3) - R^{2,1}(b_3) - R^{1,2}(a_4) + R^{1,2}(b_4) \right]$$

$$- \frac{1}{6\cdot 640}\left[-9R^{3,0}(a_5) + R^{3,0}(a_6) + 9R^{3,0}(b_5) - R^{3,0}(b_6) \right]$$

$$- \frac{1}{6\cdot 640}\left[-R^{0,3}(a_7) + 9R^{0,3}(a_8) + R^{0,3}(b_7) - 9R^{0,3}(b_8) \right]$$

$$- \frac{1}{2\cdot 960}\left[-3R^{2,1}(a_9) + 7R^{2,1}(a_{10}) + 3R^{2,1}(b_9) - 7R^{2,1}(b_{10}) \right]$$

$$\left. - \frac{1}{2\cdot 960}\left[-7R^{1,2}(a_{11}) + 3R^{1,2}(a_{12}) + 7R^{1,2}(b_{11}) - 3R^{1,2}(b_{12}) \right] \right\}.$$

Using (2.9) and (1.12) we obtain as $n \to \infty$,

$$(2n)^4\, E_n^3 \to 16\int_0^1 dt\,\frac{f^2(t)}{h^4(t)}.\left\{ \frac{1}{256}\left[-\frac{1}{3}\beta_{3,0}(t) + \frac{1}{3}\beta_{0,3}(t) + \beta_{2,1}(t) - \beta_{1,2}(t) \right] \right.$$

$$+\frac{8}{6\cdot 640}\left[\beta_{3,0}(t)-\beta_{0,3}(t)\right]-\frac{4}{2\cdot 960}\left[\beta_{2,1}(t)-\beta_{1,2}(t)\right]\Bigg\}.$$

By (1.11) we have by the symmetry of R that $R_a^{j,k}(t,t)=R_b^{k,j}(t,t)$ and thus $\beta_{j,k}(t)=-\beta_{k,j}(t)$ for $j+k=3$. Hence

$$(2n)^4 E_n^3 \rightarrow \frac{1}{120}\int_0^1 \frac{f^2(t)}{h^4(t)}\left[3\beta_{3,0}(t)+7\beta_{2,1}(t)\right] dt\,. \tag{2.21}$$

The final result follows from the expression (2.3) of the mean square error and from the asymptotics of its terms derived in (2.10), (2.13), (2.17), (2.18), (2.19) and (2.21). □

<u>PROOF OF COROLLARY 1.</u> In the weakly stationary case Part *(ii)*, of Assumption A reduces to Part (ii) of Assumption A', while Part *(iii)* is automatically satisfied. Also in this case $\beta_{3,0}(t)=-\beta_3$ and $\beta_{2,1}(t)=\beta_3$, and thus Corollary 1 follows from Theorem 2. □

<u>PROOF OF THEOREM 3.</u> The proof proceeds along lines similar to the proof of Theorem 2. Since by Assumption B.*(ii)*, $R(t,s)$ has continuous mixed partial derivatives of order four, its Taylor expansion (2.1) has the point (c_i, c_i) in place of the intermediate point int in the terms of order three and in addition it has the following fourth order terms:

$$\frac{1}{24}(t-c_i)^4 R^{4,0}(\text{ int })+\frac{1}{6}(t-c_i)^3(s-c_i)R^{3,1}(\text{ int })+\frac{1}{4}(t-c_i)^2(s-c^i)^2 R^{2,2}(\text{ int })$$

$$+\frac{1}{6}(t-c_i)(s-c_i)^3 R^{1,3}(\text{ int })+\frac{1}{24}(s-c_i)^4 R^{0,4}(\text{ int }) \tag{2.1$'$}$$

where int is again a point in the open line segment determined by (t, s) and $(c_i\, c_i)$ and depends on (t, s). Also for the Taylor expansion of $r(t)=R(t,t)$ in (2.2) we now have

$$r^{(3)}(t)=2[R^{3,0}(t,t)+3R^{2,1}(t,t)]\ \ ,\ \ r^{(4)}(t)=2[R^{4,0}(t,t)+4R^{3,1}(t,t)+3R^{2,2}(t,t)]\,, \tag{2.22}$$

and the third order term in (2.2) is modified as follows and a fourth order term is added:

$$\frac{1}{6}(t-c_i)^3 r^{(3)}(c_i)+\frac{1}{24}(t-c_i)^4\, r^{(4)}(v_t) \tag{2.2$'$}$$

where v_t is a point in between t and c_i, depending on t. When (2.1) - (2.1)′ and (2.2) - (2.2)′ are now substituted into (1.1), the resulting expression for the mean-square error is given by

$$MSE_n = E_n^{0,0} + E_n^{1,0} + E_n^{2,0} + E_n^{1,1} + (E_n^r)' + (E_n^3)' + E_n^4$$

where the first four terms on the right side are given by (2.3.1) - (2.3.4), $(E_n^r)'$ and $(E_n^3)'$ are modifications of (2.3.5) and (2.3.6) respectively and E_n^4 is a new term. We have

$$(E_n^r)' = \sum_{i=1}^n \frac{1}{12}\Delta t_i \left\{ r^{(3)}(c_i) \int_{A_i} (t-c_i)^3 f^2(t)dt + \frac{1}{4} \int_{A_i} (t-c_i)^4 f^2(t) r^{(4)}(v_t)\, dt \right\}, \qquad (2.3.5)'$$

the term $(E_n^3)'$ modifies E_n^3 of (2.3.6) and simplies to

$$(E_n^3)' = -\sum_{i=1}^n R^{3,0}(c_i, c_i) \frac{1}{6} \int_{A_i}\int_{A_i} [(t-c_i)^3 + (s-c_i)^3] f(t) f(s)\, dt\, ds\,,$$

$$-\sum_{i=1}^n R^{2,1}(c_i, c_i) \frac{1}{2} \int_{A_i}\int_{A_i} [(t-c_i)^2(s-c_i) + (t-c_i)(s-c_i)^2] f(t) f(s)\, dt\, ds\,, \qquad (2.3.6)'$$

because the term in (2.3.6) involving $\int_{A_i}$ is identically zero once $R^{j,k}(u_t, 2c_i - u_t)$ is replaced by $R^{j,k}(c_i, c_i)$. The additional fourth order term E_n^4 is given by

$$E_n^4 = \sum_{i=1}^n \frac{1}{2} \Delta t_i \int_{A_i} f(t) f(2c_i - t)(t-c_i)^4 \left[\frac{1}{24} R^{4,0} - \frac{1}{6} R^{3,1} + \frac{1}{4} R^{2,2} - \frac{1}{6} R^{1,3} + \frac{1}{24} R^{0,4}\right] (u_t, 2c_i - u_t)\, dt$$

$$-\sum_{i=1}^n \int_{A_i}\int_{A_i} f(t) f(s) \Big[\frac{1}{24}(t-c_i)^4 R^{4,0}(\text{ int }) + \frac{1}{6}(t-c_i)^3(s-c_i) R^{3,1}(\text{ int }) + \frac{1}{4}(t-c_i)^2(s-c_i)^2 R^{2,2}(\text{ int })$$

$$+ \frac{1}{6}(t-c_i)(s-c_i)^3 R^{1,3}(\text{ int }) + \frac{1}{24}(s-c_i)^4 R^{0,4}(\text{ int })\Big]\, dt\, ds \qquad (2.3.7)$$

where u_t is a point in between t and c_i and " int " is a point in the open line segment of (s, t) and (c_i, c_i).

The asymptotics of the terms (2.3.1) - (2.3.4) are given in (2.10), (2.13), (2.17) and (2.18).

The term $(E_n^r)'$. From the Taylor expansion (2.4) of f we find

$$\int_{A_i} (t-c_i)^3 f^2(t)\, dt = \Delta t_i^5 \frac{1}{40} f(c_i) f'(c_i) + o(n^{-5})\,,$$

$$\int_{A_i} (t-c_i)^4 f^2(t)\, dt = \Delta t_i^5 \frac{1}{80} f^2(c_i) + o(n^{-5})\,,$$

where the $o(n^{-5})$ terms are uniform in i. Substituting into (2.3.5)' we obtain

$$(E_n^r)' = \sum_{i=1}^{n} \left\{ \Delta t_i^6 \, \frac{1}{960} \left[2f(c_i) f'(c_i) r^{(3)}(c_i) + \frac{1}{4} f^2(c_i) r^{(4)}(\text{ int }) \right] + o(n^{-6}) \right\}$$

and using (2.9) we have as $n \to \infty$,

$$(2n)^5 \, (E_n^r)' \to \frac{1}{30} \int_0^1 [2f(t) f'(t) r^{(3)}(t) + \frac{1}{4} f^2(t) r^{(4)}(t)] \, \frac{dt}{h^5(t)} \, . \tag{2.23}$$

<u>The term $(E_n^3)'$.</u> Using (2.5), (2.11), (2.14) and

$$\int_{A_i} (t - c_i)^3 f(t) \, dt = \Delta t_i^5 \, \frac{1}{80} f'(c_i) + \Delta t_i^6 \, \frac{1}{768} \, [f''(\text{ int}_{11}) - f''(\text{ int }_{10})] \, , \tag{2.24}$$

we obtain

$$(E_n^3)' = -\sum_{i=1}^{n} R^{3,0}(c_i c_i) \left\{ \Delta t_i^6 \, \frac{1}{240} f(c_i) f'(c_i) + o(n^{-6}) \right\} - \sum_{i=1}^{n} R^{2,1}(c_i \; c_i) \left\{ \Delta t_i^6 \, \frac{1}{144} f(c_i) f'(c_i) + o(n^{-6}) \right\}$$

where the $o(n^{-6})$ terms are uniform in i. It then follows by (2.9) that as $n \to \infty$,

$$(2n)^5 \, (E_n^3)' \to -\int_0^1 \frac{f(t) f'(t)}{h^5(t)} \left[\frac{2}{15} R^{3,0}(t, t) + \frac{2}{9} R^{2,1}(t, t) \right] dt \, . \tag{2.25}$$

<u>The term E_n^4.</u> When the Taylor expansion (2.4) of f is substituted in (2.3.7), the dominant term corresponds to $f(c_i)$ and we obtain

$$E_n^4 = \sum_{i=1}^{n} [f^2(c_i) + o(1)] \times \tag{2.26}$$

$$\times \left\{ \frac{1}{2} \Delta t_i \int_{A_i} (t - c_i)^4 \left[\frac{1}{24} R^{4,0} - \frac{1}{6} R^{3,1} + \frac{1}{4} R^{2,2} - \frac{1}{6} R^{1,3} + \frac{1}{24} R^{0,4} \right] (u_t, \, 2c_i - u_t) \, dt \right. \tag{2.26.1}$$

$$- \iint_{A_i A_i} \left[\frac{1}{24} (t - c_i)^4 R^{4,0}(\text{ int }) + \frac{1}{24} (s - c_i)^4 R^{0,4}(\text{ int }) + \frac{1}{4} (t - c_i)^2 (s - c_i)^2 R^{2,2}(\text{ int }) \right] dt \, ds$$

(2.26.2)

$$\left. - \iint_{A_i A_i} \left[\frac{1}{6} (t - c_i)^3 (s - c_i) R^{3,1}(\text{ int }) + \frac{1}{6} (t - c_i)(s - c_i)^3 R^{1,3}(\text{ int }) \right] \, ds \, ds \right\} \tag{2.26.3}$$

where o(1) is uniform in i. Applying the mean value theorem, we find that the term (2.26.1) equals

$$\Delta t_i^6 \frac{1}{2\cdot 80\cdot 24}\left[R^{4,0}+R^{0,4}-4R^{3,1}-4R^{1,3}+6R^{2,2}\right](\mathrm{int}_1), \qquad (2.26.1)'$$

and the term (2.26.2) equals

$$-\Delta t_i^6 \frac{1}{3\cdot 80\cdot 24}\left[3R^{4,0}(\mathrm{int}_2)+3R^{0,4}(\mathrm{int}_3)+10R^{2,2}(\mathrm{int}_4)\right]. \qquad (2.26.2)'$$

In order to apply the mean value theorem in the terms of (2.26.3), the square $A_i \times A_i$ is split into its four squares with half size over each of which $(t-c_i)(s-c_i)$ has constant sign. We thus find

$$\int_{A_i}\int_{A_i}(t-c_i)^3(s-c_i)R^{3,1}(\mathrm{int})\,dt\,ds =$$

$$=\left\{R^{3,1}(\mathrm{int}_5)\int_{t_{i-1}}^{c_i}\int_{t_{i-1}}^{c_i}+R^{3,1}(\mathrm{int}_6)\int_{c_i}^{t_i}\int_{t_{i-1}}^{c_i}+R^{3,1}(\mathrm{int}_7)\int_{c_i}^{t_i}\int_{c_i}^{t_i}+R^{3,1}(\mathrm{int}_8)\int_{t_{i-1}}^{c_i}\int_{c_i}^{t_i}\right\}(t-c_i)^3(s-c_i)\,dt\,ds$$

$$=\Delta t_i^6\frac{1}{512}\left\{R^{3,1}(\mathrm{int}_5)-R^{3,1}(\mathrm{int}_6)+R^{3,1}(\mathrm{int}_7)-R^{3,1}(\mathrm{int}_8)\right\} \qquad (2.27)$$

where int_i are points in $A_i \times A_i$, and likewise for the other term. It then follows from (2.26), (2.27) and (2.9) that as $n \to \infty$,

$$\begin{aligned}(2n)^5 E_n^4 \to 2^5\int_0^1 dt\,\frac{f^2(t)}{h^5(t)}\Bigg\{&\frac{1}{2\cdot 80\cdot 24}\left[R^{4,0}+R^{0,4}-4R^{3,1}-4R^{1,3}+6R^{2,2}\right](t,t)\\ &-\frac{1}{3\cdot 80\cdot 24}\left[3R^{4,0}+3R^{0,4}+10R^{2,2}\right](t,t)\\ &-\frac{1}{6\cdot 512}\left[2R^{3,1}-2R^{3,1}\right](t,t)-\frac{1}{6\cdot 512}\left[2R^{1,3}-2R^{1,3}\right](t,t)\Bigg\}\end{aligned}$$

$$=-\frac{1}{60}\int_0^1\frac{f^2(t)}{h^5(t)}\left[R^{4,0}(t,t)+4R^{3,1}(t,t)+\frac{1}{3}R^{2,2}t,t)\right]dt. \qquad (2.28)$$

Finally, adding (2.10), (2.13), (2.17), (2.18), (2.23), (2.25) and (2.28), we obtain (1.17) and (1.18). The final compact expression in (1.18) follows by straightforward calculation. □

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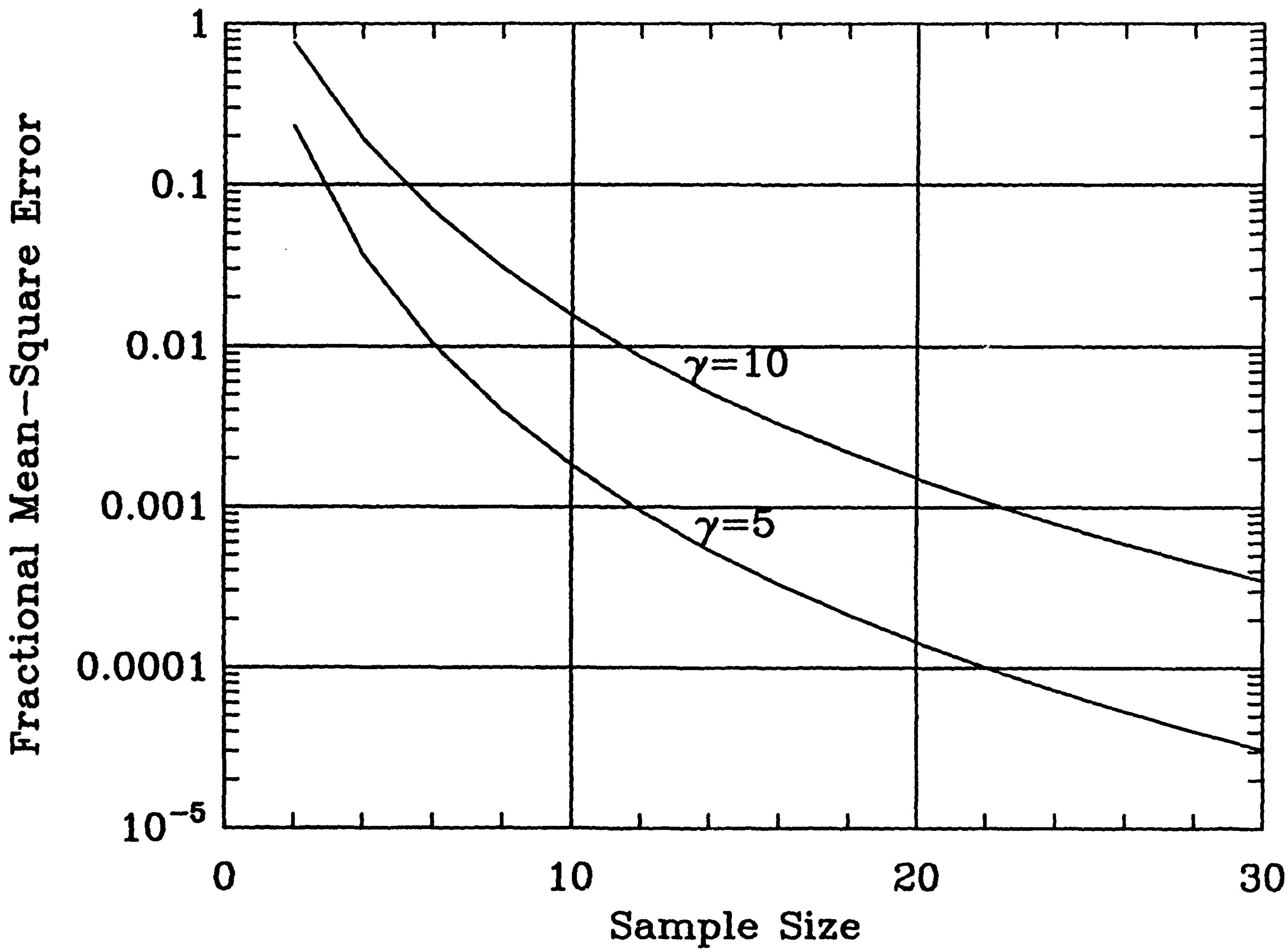

Fig. 1 Fractional mean-square error $mse(N)/\sigma^2$ as a function of the sample size N.

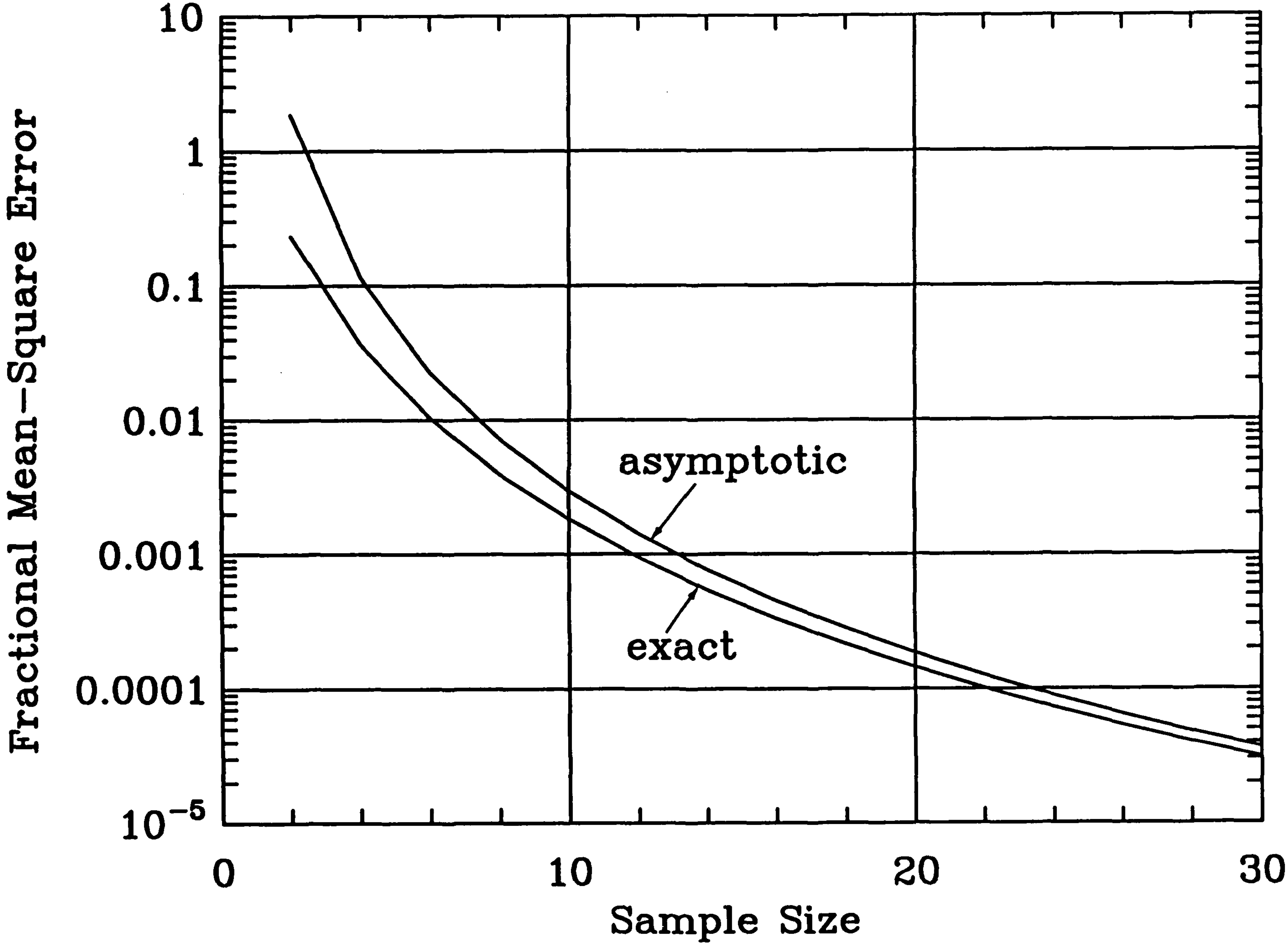

Fig. 2 Exact and asymptotic fractional mean-square errors as functions of the sample size N (γ=5).

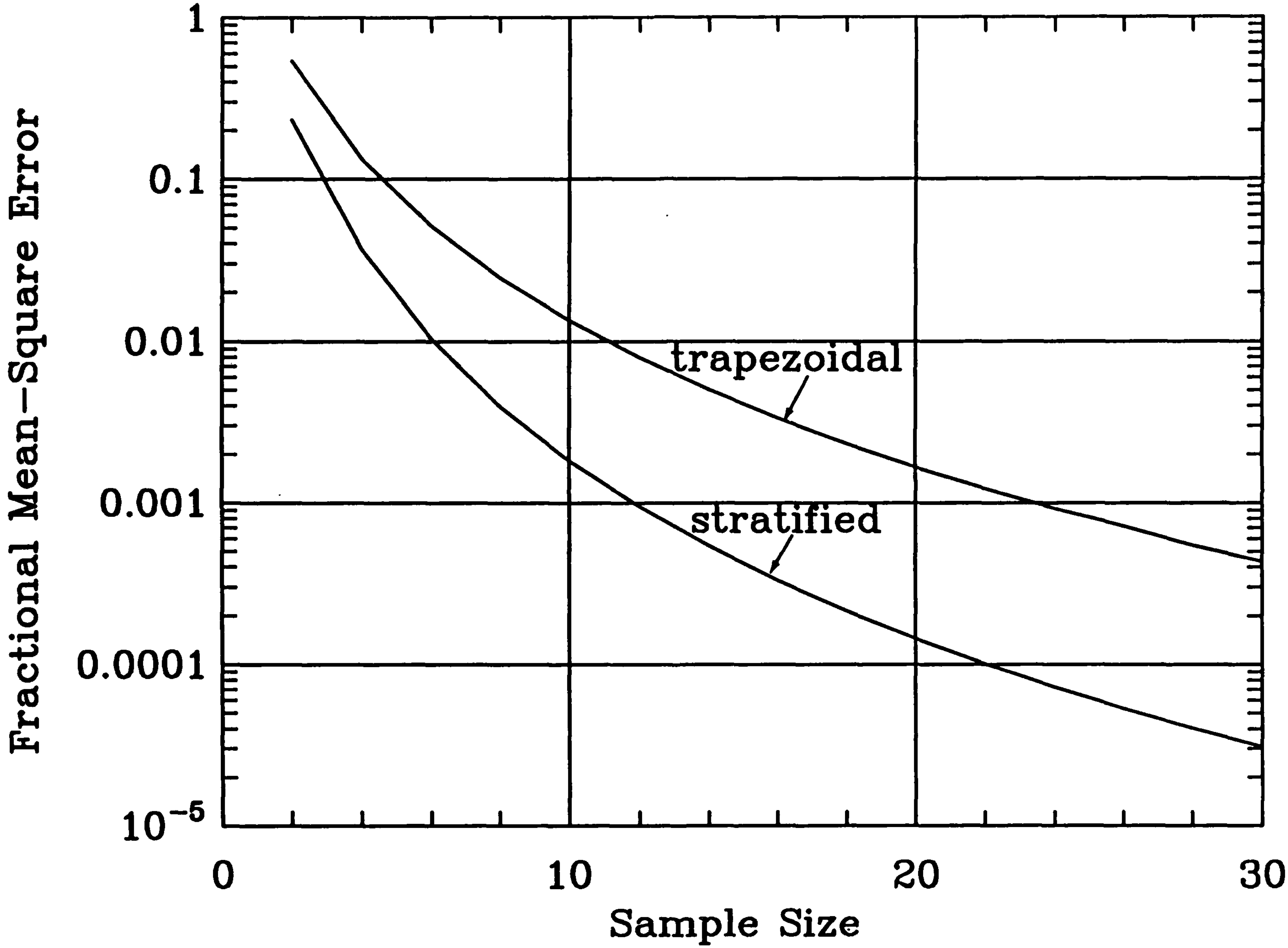

Fig. 3 Fractional mean-square errors of the stratified and trapezoidal approximations as functions of the sample size N (γ=5).

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